## LECTURE 30 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

**Example.** We check the following limits.

(1) $\lim_{x\to 0} \frac{3x-\sin x}{x} \overset{\text{“}\frac{0}{0}\text{”, L'H}}{=} \lim_{x\to 0} \frac{3-\cos(x)}{1} = 2.$

(2) $\lim_{x\to 0} \frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} \overset{\text{“}\frac{0}{0}\text{”, L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \overset{\text{“}\frac{0}{0}\text{”, L'H}}{=} \lim_{x\to 0} \frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} = -\frac{1}{8}.$

**Example.** One may be tempted to applied L'Hôpital's Rule everywhere without checking the hypothesis.

$$\lim_{x\to 0} \frac{1-\cos(x)}{x+x^2} \overset{\text{“}\frac{0}{0}\text{”, L'H}}{=} \lim_{x\to 0} \frac{\sin(x)}{1+x} = 0$$

where at the last inequality, you want to apply L'Hôpital's Rule again but notice that $\sin(0) = 0$ but the denominator yields 1 by directly plugging in.

*Remark.* L'Hôpital's Rule applies to one-sided limits as well.

We are not going to fully justify how L'Hôpital's Rule will apply also to indeterminate forms other than "$\frac{0}{0}$".

**Example.** Find the limits of these "$\frac{\infty}{\infty}$" forms:

(1) $\lim_{x\to\frac{\pi}{2}} \frac{\sec(x)}{1+\tan(x)}$.
Note that the numerator and the denominator are discontinuous at $x = \frac{\pi}{2}$. We then must check one-sided limits.

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\sec(x)}{1+\tan(x)} \overset{\text{“}\frac{\infty}{\infty}\text{”, L'H}}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{\sec(x)\tan(x)}{\sec^2(x)} = \lim_{x\to\frac{\pi}{2}^-} \sin(x) = 1,$$

and

$$\lim_{x\to\frac{\pi}{2}^+} \frac{\sec(x)}{1+\tan(x)} \overset{\text{“}\frac{-\infty}{-\infty}\text{”, L'H}}{=} \lim_{x\to\frac{\pi}{2}^+} \frac{\sec(x)\tan(x)}{\sec^2(x)} = \lim_{x\to\frac{\pi}{2}^+} \sin(x) = 1.$$

(2) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} \overset{\text{“}\frac{\infty}{\infty}\text{”, L'H}}{=} \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$. This implies that logarithmic growth is slower than $x^{\frac{1}{2}}$ (in fact, slower than $x^a$ for **any** $a > 0$).

(3) $\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{“}\frac{\infty}{\infty}\text{”, L'H}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{“}\frac{\infty}{\infty}\text{”, L'H}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$. This implies that exponential growth is faster than $x^2$ (in fact, faster than $x^a$ for **any** $a > 0$).

**Example.** Find the limits of these $\infty \cdot 0$ forms:

(1) $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) \overset{y=\frac{1}{x}}{=} \lim_{y\to 0^+} \frac{\sin(y)}{y} = 1$. The last step is either by L'Hôpital's Rule or the limit identity you remembered (proved by the squeeze theorem).

(2) $\lim_{x\to 0^+} \sqrt{x}\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/\sqrt{x}} \overset{\text{“}\frac{0}{0}\text{”, L'H}}{=} \lim_{x\to 0^+} -\frac{1/x}{1/2x^{\frac{3}{2}}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$. Note the initial indeterminante form is "$0\cdot(-\infty)$", but we transformed it into a "$\frac{0}{0}$". This limit implies that even though $\ln(x) \to -\infty$ as $x \to 0$, multiplying by a growth term such as $\sqrt{x}$ will steer it back to 0, instead of diving down. In fact, $\lim_{x\to 0^+} x^a \ln(x) = 0$ for **any** $a > 0$.

**Example.** Find the limit of this $\infty - \infty$ form:

$$\lim_{x\to 0} \frac{1}{\sin(x)} - \frac{1}{x} = \lim_{x\to 0} \frac{x-\sin(x)}{x\sin(x)} \overset{\text{“}\frac{0}{0}\text{”, L'H}}{=} \lim_{x\to 0} \frac{1-\cos(x)}{\sin(x)+x\cos(x)} \overset{\text{“}\frac{0}{0}\text{”, L'H}}{=} \lim_{x\to 0} \frac{-\sin(x)}{2\cos(x)+x\cos(x)} = \frac{0}{2} = 0.$$

**Example.** Logarithmic tricks: if $\lim_{x\to a} \ln(f(x)) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to 0} e^{\ln f(x)} = e^{(\lim_{x\to 0} \ln f(x))} = e^L.$$

With this, we can prove that $\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = e$. Now, consider the function $f(x) = (1+x)^{\frac{1}{x}}$ and we want $\lim_{x\to 0^+} f(x)$. By the logarithmic trick, we simply need

$$\lim_{x\to 0^+} \ln(f(x)) = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \overset{\text{“}\frac{\infty}{\infty}\text{”, L'H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = 1.$$

Therefore, by the trick,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^{\left(\lim_{x\to 0^+} \ln f(x)\right)} = e.$$

**Example.** Similar example. Fiund$\lim_{x\to\infty} x^{\frac{1}{x}}$. We use the trick again. Let $f(x) = x^{\frac{1}{x}}$. We just need to find $\lim_{x\to\infty} \ln(f(x))$.

$$\lim_{x\to\infty} \ln(f(x)) = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{“}\frac{\infty}{\infty}\text{”, L'H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Therefore,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^{(\lim_{x\to\infty} \ln f(x))} = e^0 = 1.$$

Now, armed with another way to evaluate limits (under some specific conditions), we are now able to deal with a bit more complicated problems.

**Example.** Find a value of $c$ that makes the function

$$f(x) = \begin{cases} \frac{9x - 3\sin(3x)}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of $c$ works.

We want the left and right limit approaching $x = 0$ to be equal to the function value $f(0) = c$, that is, we want

$$\lim_{x\to 0^-} \frac{9x - 3\sin(3x)}{5x^3} = c,$$

and

$$\lim_{x\to 0^+} \frac{9x - 3\sin(3x)}{5x^3} = c.$$

It suffices to evaluate both limits.

$$\lim_{x\to 0^-} \frac{9x - 3\sin(3x)}{5x^3} \overset{\text{“}\frac{0}{0}\text{”, L'H}}{=} \lim_{x\to 0^-} \frac{9 - 9\cos(3x)}{15x^2} \overset{\text{“}\frac{0}{0}\text{”, L'H}}{=} \lim_{x\to 0^-} \frac{27\sin(3x)}{30x} \overset{\text{“}\frac{0}{0}\text{”, L'H}}{=} \frac{27}{30} \lim_{x\to 0^-} \frac{3\cos(3x)}{1} = \frac{27}{10}.$$

The right limit is similar. We then arrive at $c = \frac{27}{10}$.